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Local classification of varieties in the symplectic space

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1 Introduction.

In this note we will provide a survey on several recent results on the local classification problem of varieties under symplectomorphisms.

In general, there are two types of local classification problem: (V) the classification of mappings and varieties, and (D) the classification of differential forms and dynamical systems.

As a general tendency of results, for the classification problem of type (V), we have *finite lists* for simplest objects after classification and at most *finite dimensional moduli* for complicated objects. Finite determinacy holds for objects except for infinite codimensional set of objects. Then the C^∞ classification and the analytic classification have no essential difference on the classification results. In fact, the classification of isotropic or Lagrangian varieties or mappings under symplectomorphisms turns to fall into type (V), and several finiteness theorems are proved for them [9][10][12]. Note that the differential classification of mappings under the right-left equivalence or $\mathcal{A}$-equivalence in the sense of Mather belongs to (V) of course ([15]). We observe that the differential and symplectic classifications coincide for map-germs for simplest objects (1st stage). Then, for more complicated singularities, there appears a difference between differential and symplectic classifications, the existence of "ghosts" in the sense of Arnold [2] (2nd stage). The differential classification has a finite list while the symplectic classification has finite dimensional moduli ([11][13]). Moreover, if we proceed to more complicated objects further, then we have finite dimensional moduli for both differential and symplectic classifications (3rd stage).

On the other hand, for the classification of type (D), we have finite lists for simplest objects after classification at first (1st stage). In the 1-st stage, problems from (V) and (D) look similar and no difference between (V) and (D) is observed after classification. Then we have still finite lists more complicated objects under the C^∞ classification, while we have *functional moduli* for the analytic classification (2nd stage). If we proceed to more complicated objects further, then we have functional moduli for both C^∞ and analytic classification (3rd stage).

The 2nd stage can be phrased as a "time-lag" of classification. The existence of a "time-lag" depends on cases in the classification problems.

In this talk, we observe that a classification problem of (non-Lagrangian) coisotropic mappings falls into type (D). Therefore there is clear difference between differential and symplectic classifications. Moreover we investigate the "time-lag" for generic symplectic classification of map-germs $\mathbb{R}^3 \to \mathbb{R}^4$.

2 symplectic classification of map-germs.

Let ω be a symplectic form on $\mathbb{R}^{2n}$, and $f : (\mathbb{R}^m, a) \to \mathbb{R}^{2n}$ a C^∞ map-germ. We consider the classification problem of the pair (f, ω) fixing m and n: The pair (f, ω) is called *symplectomorphic* to another pair (f', ω') if there exist a diffeomorphism-germ $\sigma : (\mathbb{R}^m, a) \to (\mathbb{R}^m, a')$ and a symplectomorphism-germ $\tau : (\mathbb{R}^{2n}, f(a)) \to (\mathbb{R}^{2n}, f'(a'))$, $\tau^*\omega' = \omega$, such that $f' \circ \sigma = \tau \circ f$, namely that the diagram

$$\begin{array}{ccc} (\mathbb{R}^m, a) & \xrightarrow{f} & (\mathbb{R}^{2n}, f(a)),\ \omega \\ \sigma \downarrow & & \downarrow \tau \\ (\mathbb{R}^m, a') & \xrightarrow{f'} & (\mathbb{R}^{2n}, f'(a')),\ \omega' \end{array}$$

commutes.

If the above condition is satisfied just for a diffeomorphism-germ τ, (not necessarily a symplectomorphism-germ), then we call f and f' are *diffeomorphic*.

First we mention a theorem which contains the classical Darboux theorem as the special case $m = 0$:

Theorem 1 (Darboux-Givental [4]) *For any immersion-germs $f, f' : \mathbb{R}^m \to \mathbb{R}^{2n}$ and for any symplectic forms ω, ω' on $\mathbb{R}^{2n}$, (f, ω) and (f', ω') are symplectomorphic if and only if two forms $f^*\omega$ and $f'^*\omega'$ are diffeomorphic; for some diffeomorphism-germ σ on $\mathbb{R}^m$, $\sigma^*(f'^*\omega') = f^*\omega$.*

Thus in the non-singular case (the case of immersion-germs), the classification problem is reduced to that of pull-back forms to the sources. Note that the pull-backs of symplectic forms are not arbitrary. In particular we have

Corollary 2 *All non-singular hypersurface-germs in $\mathbb{R}^{2n}$ are symplectomorphic.*
All coisotropic (resp. isotropic) submanifold-germs of fixed dimension in $\mathbb{R}^{2n}$ are symplectomorphic.

Note that all immersion-germs (on a fixed dimensional source) are diffeomorphic in our sense. In the singular case, however, even if f and f' are diffeomorphic and $f^*\omega$ and $f'^*\omega'$ are diffeomorphic, (f,ω) and (f',ω') are not necessarily symplectomorphic.

In fact, in the case $m = n = 1$ (planer curves), we have given both symplectic and differential exact classifications of differentially uni-modal plane curve singularities, and clarified the difference of differential and symplectic classifications ([11][13]). For the classification of curves ($m = 1, n \geq 2$), see [2][3][14][7][6].

3 Classification of isotropic surfaces.

A pair (f,ω) is called *isotropic* if $f^*\omega = 0$. Then f is called *isotropic* with respect to ω. If $m = 1$, then any pair (f,ω) is isotropic. Moreover if $f : \mathbb{R}^n \to \mathbb{R}^{2n}$ then we call (f,ω) *Lagrangian.*

In the case $m = n = 2$, we have

Theorem 3 ([9]) *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, \omega)$ be isotropic. Suppose f is diffeomorphic to*

$$f_{\mathrm{ou}}(t,u) = (ut, t^2, \frac{2}{3}t^3, u) = (p_1, q_1, p_2, q_2).$$

Then, for any symplectic form ω, the pair (f,ω) is symplectomorphic to $(f_{\mathrm{ou}}, \omega_{\mathrm{st}})$. (Darboux-type theorem). *Moreover, for* *any n, there exists a class of open umbrellas, characterised by the symplectically structurally stability, and for them, Darboux type theorem holds.*

We refer to a generalization of Darboux-Givental case to singular case.

Theorem 4 (Domitrz, Janeczko, Zhitomirskii, [7], 2006) *For any $N, N' \subset \mathbb{R}^{2n}$ quasi-homogeneous, for any symplectic forms ω, ω' on $\mathbb{R}^{2n}$, (N, ω) and (N', ω') are symplectomorphic if and only if their algebraic restrictions $[\omega]_N$ and $[\omega']_{N'}$ are diffeomorphic.*

Corollary 5 *Algebraic restrictions of symplectic forms to an open umbrella are diffeomorphic to each other.*

Example 6 Let $f_\lambda(u,t) := (t^5 + ut^3 + \lambda u^2 t,\ t^2,\ \frac{2}{5}t^5 + \frac{4}{3}\lambda ut^3,\ u) = (p_1, q_1, p_2, q_2)$, $\lambda \neq \frac{21}{100}$. Then the family f_λ of isotropic map-germs with respect to ω_{st} is trivialised by diffeomorphisms, but λ gives the "symplectic moduli".

There is the notion of *symplectic codimension* $\text{sp-codim}(f, \omega)$ also for an isotropic pair (f, ω). The number $\text{sp-codim}(f, \omega)$ is characterised as the minimal number of symplectically versal unfolding of f.

Theorem 7 ([12]) $\text{sp-codim}(f, \omega)$ *is a diffeomorphism invariant for isotropic nomalisations $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{2n}, \omega)$: If f and f' are diffeomorphic, then* $\text{sp-codim}(f, \omega) = \text{sp-codim}(f', \omega')$ *for any symplectic forms ω, ω' with $f^*\omega = 0, f'^*\omega' = 0$.*

In the complex analytic case, if $\text{codim}\Sigma(f) \geqq 2$, then

$$\text{sp-codim}(f, \omega) = \dim_{\mathbb{C}} \mathcal{R}_f / f^* \mathcal{O}_{2n},$$

where

$$\mathcal{R}_f := \{h \in \mathcal{O}_n \mid dh \in \mathcal{O}_n \cdot df\}.$$

In the case $n = 1$, we have

$$\text{sp-codim}(f, \omega) = \dim_{\mathbb{C}} \mathcal{O}_1 / f^* \mathcal{O}_2.$$

Moreover the difference of differential/symplectic classification is given by

$$\text{gh}(f, \omega) := \dim_{\mathbb{C}} \mathcal{G}_f / f^* \mathcal{O}_{2n},$$

symplectic defect or *ghost number* where

$$\mathcal{G}_f := \{h \in \mathcal{O}_n \mid dh \in f^* \Omega^1_{2n}\} = \{h \in \mathcal{O}_n \mid dh \in f^* \mathcal{O}_{2n} \cdot df\}.$$

Remark that

$$\mathcal{R}_f \supseteq \mathcal{G}_f \supseteq f^* \mathcal{O}_{2n}, \qquad f^* : \mathcal{O}_n \leftarrow \mathcal{O}_{2n}.$$

Example 8 For the open umbrella

$$f_{\text{ou}} = (ut, t^2, \frac{2}{3} t^3, u) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^4, 0),$$

we have that

$$dh(t, u) \in \langle d(t^2), du, d(ut), d(\frac{2}{3} t^3) \rangle_{\mathcal{O}_2} = \langle tdt, du, udt \rangle_{\mathcal{O}_2}$$

if and only if $h = a(t^2, t^3, ut, u)$ for some C^∞ function a. Therefore $\mathcal{R}_f = \mathcal{G}_f = f^* \mathcal{O}_4$.

Proposition 9 *Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be isotropic map-germ of corank ≤ 1 for a symplectic form ω. If* sp-codim$(f, \omega) \leq 1$ *then (f, ω) is symplectomorphic to $(f_{\text{ou}}, \omega_{\text{st}})$ the open umbrella, or to $(f^{\pm}_{\text{mou}}, \omega_{\text{st}})$ the multiple open umbrella, where $f^{\pm}_{\text{mou}}(t, u) := (t^3 \pm u^2 t, t^2, \frac{4}{3} u t^3, u)$.*

Moreover $(f^{+}_{\text{mou}}, \omega_{\text{st}})$ is not symplectomorphic to $(f^{-}_{\text{mou}}, \omega_{\text{st}})$. In fact f^{+}_{mou} and f^{-}_{mou} are not diffeomorphic.

Remark 10 For the multiple open umbrella, $\mathcal{R}_f \supsetneqq \mathcal{G}_f = f^*\mathcal{O}_{2n}$: There is no ghost in this case. The map-germs f_λ in Example 6, we have that sp-codim$(f_\lambda, \omega_{\rm st}) = 2$, and that $\mathcal{R}_{f_\lambda} \supsetneqq \mathcal{G}_{f_\lambda} \supsetneqq {f_\lambda}^*\mathcal{O}_{2n}$.

4 Symplectic classification of Whitney umbrellas.

Now we consider the symplectic classification of generic map-germs $f : \mathbb{R}^3 \to \mathbb{R}^4$ as a typical example of our classification problem.

As for the differential classification, it is known that a generic map-germ $f : \mathbb{R}^3 \to \mathbb{R}^4$ is diffeomorphic to an immersion or to a Whitney umbrella. A map-germ $f : (\mathbb{R}^3, a) \to \mathbb{R}^4$ is called a *Whitney umbrella* if f is diffeomorphic to the map-germ $(\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ given by $(u, v, w) \mapsto (p_1, q_1, p_2, q_2) = (uv, u^2, w, v)$.

The double point locus $D(f)$ (resp. singular point locus $S(f)$) of the (normalized) Whitney umbrella, designated also as f, is given by $\{v = 0\}$ (resp. $\{u = v = 0\}$). In fact, the points $(\pm u, 0, w)$ are mapped to the same point by f. Thus we have the *canonical stratification* of $\mathbb{R}^3$ associated to f : $\mathbb{R}^3 \supset D(f) \supset S(f)$. Moreover note that the kernel field $K(f)$ of the differential $f_* : T\mathbb{R}^3 \to T\mathbb{R}^4$ along $S(f) = \{u = v = 0\}$ is given by $K(f)(0, 0, w) = \frac{\partial}{\partial u}$.

On the other hand, for a generic symplectic form ω, the pull-back $f^*\omega$ on $\mathbb{R}^3$ is of rank 2. Then the kernel field of $f^*\omega$ is called the *characteristic filed* of (f, ω) and we have the *characteristic foliation* $\mathcal{F} = \mathcal{F}_{(f,\omega)}$ on $\mathbb{R}^3$. The relative position of the characteristic foliation of (f, ω) and the canonical stratification of f is clearly an symplectically invariant character of (f, ω).

For example, the standard symplectic form $\omega_{\rm st} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, pulled back by f,

$$f^*\omega_{\rm st} = d(uv) \wedge d(u^2) + dw \wedge dv = d(w - \frac{2}{3}u^3) \wedge dv$$

is of rank 2. In this example, the characteristic foliation is given by $w - \frac{2}{3}u^3 = $ const., $v =$ const.. Therefore each characteristic curve is contained in the singular locus $S(f) = \{v = 0\}$, and that situation is never generic.

Note that the kernel field $K(f)$ of the differential f_* coincides with the characteristic field along $S(f)$. Hence each characteristic curve is necessarily tangent to the locus $D(f)$ of double points along $S(f)$.

Generically, each characteristic curve contacts with the double point locus $D(f)$ in the second order along $S(f)$ except isolated points of $S(f)$, and, in the third order at those isolated points.

Define $g : \mathbb{R}^3 \to \mathbb{R}^2$ as the symplectic reduction determined by the characteristic foliation of f (which is determined up to left equivalence). Consider the map $g|_{D(f)} : D(f) \to \mathbb{R}^2$. If each characteristic curve contacts with the double point locus $D(f)$ in the second order along $S(f)$, then $g|_{D(f)}$ has a fold singularity along $S(f)$ and it is a two-to-one mapping off $S(f)$, which induces an involution $\tau(f) : D(f) \to D(f)$ on the surface $D(f)$. Moreover, $f|_{D(f)} : D(f) \to \mathbb{R}^3$ is also two-to-one off $S(f)$. It also induces an involution $\eta(f) : D(f) \to D(f)$ on $D(f)$. So we have a *pair of involutions* $(\tau(f), \eta(f))$ on the surface $D(f)$. If a characteristic curve contacts with $D(f)$ in the third order at a point $S(f)$, then $g|_{D(f)} : D(f) \to \mathbb{R}^2$ has a more degenerate singularity than the fold singularity.

Similar situation appeared in the classification of glancing hypersurface due to Melrose [16][17]. See also [1][20].

Consider (not a mono-germ but) a bi-germ $f = f_1 \coprod f_2$: $(\mathbb{R}^3, 0) \coprod (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ and the standard symplectic form ω_{st} on $(\mathbb{R}^4, 0)$. Suppose f_1 and f_2 are transversal immersion-germs. Then the self-intersection forms a smooth surface S in $(\mathbb{R}^4, 0)$. Consider the characteristic foliations $\mathcal{F}_1$ on $M_1 = f_1(\mathbb{R}^3, 0)$ and $\mathcal{F}_2$ on $M_2 = f_2(\mathbb{R}^3, 0)$. Then the relative position of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to S is a symplectically invariant character. If $\mathcal{F}_1$ is transversal to S in M_1, then $\mathcal{F}_2$ is transversal to S in M_2. Then the pair is symplectomorphic to the standard one: $M_1 = \{p_1 = 0\}$ and $M_2 = \{q_1 = 0\}$.

M_1 and M_2 are said to be *glancing* at a point in S if the both characteristic curve through the point is tangent to S in the second order [16]. Generically M_1 and M_2 are glancing along a smooth curve in S and at isolated points the tangency becomes of higher order.

Melrose [16] showed that any glancing pair is C^∞ symplectomorphic to the pair $\{p_1 = p_2^2\}$ and $\{q_2 = 0\}$. On the other hand, in [19], Oshima gave a counter example to the uniqueness result for the analytic classification. (A counter example to Sato's conjecture [18]). In fact it is known that the analytic symplectic classification of glancing pairs has a functional moduli.

Actually we announce the following result:

Theorem 11 *For a generic pair (f, ω) of a C^∞ mapping f : $\mathbb{R}^3 \to \mathbb{R}^4$ and a C^∞ symplectic form ω, at any singular point $a \in \mathbb{R}^3$ of f, (f, ω) is symplectomorphic to the normal form*

$$\omega_1 = dp_1 \wedge dq_1 + dp_2 \wedge d(q_2 - q_1),$$

or to

$$\omega_2 = dp_1 \wedge dq_1 + dp_2 \wedge d(q_2 - q_1p_2 - \varphi(q_1^2)),$$

for a functional moduli $\varphi, (\varphi(0) = \varphi'(0) = 0)$ *with the normal form* $(u, v, w) \mapsto (p_1, q_1, p_2, q_2) = (uv, u^2, w, v)$.

Note that, for the normal forms in Theorem 11, the pull-back form turns out to be

$$d(w - \frac{2}{3}u^3) \wedge d(v - u^2), \text{ or } d(w - \frac{2}{3}u^3) \wedge d(v - u^2 w + \frac{2}{5}u^5 - \varphi(u^2)),$$

Remark 12 *There appears a difference between* C^∞ *and analytic classification in Theorem 11 arising from the conjugate classification of map-germs* $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *with* 3*-jets of type* $(u, w) \to (u, w + u^3)$*: In the sense of Voronin, the* $\mathcal{B}_3$*-classification problem arises. In fact the composition* $\eta(f) \circ \tau(f) : D(f) \to D(f)$ *is of this form. Remark that the symplectic classification of swallowtails corresponds to the* $\mathcal{B}_5$*-classification problem.*

Remark 13 The above classification is also regarded as the classification of *coisotropic pairs.* A pair (f, ω) of a map-germ $f : (\mathbb{R}^m, 0) \to \mathbb{R}^{2n}$ and a symplectic form-germ ω on $\mathbb{R}^{2n}$, $(m \geq n)$, is called *coisotropic* if f lifts to an isotropic map-germ $\widetilde{f} : (\mathbb{R}^m, 0) \to (\mathbb{R}^{2m}, 0) = (\mathbb{R}^{2n}, 0) \times (\mathbb{R}^{2(m-n)}, 0)$ with a symplectic form $\pi_1^*\omega - \pi_2^*\mu$.

Any coisotropic immersion $(\mathbb{R}^m, 0) \to (\mathbb{R}^{2n}, 0)$ for any symplectic form, in the ordinary sense, ω lifts to an Lagrangian immersion into $\mathbb{R}^{2m}$, so coisotropic in the above sense.

Then we define the *symplectic codimension* of coisotropic pair (f, ω) by

$$\text{sp-codim}(f, \omega) := \dim_{\mathbb{R}} \mathcal{R}_f / (f^*\mathcal{O}_{2n} + g^*\mathcal{O}_{2(m-n)}).$$

For normal forms we have

$$\text{sp-codim}(f, \omega_1) = 0$$

and

$$\text{sp-codim}(f, \omega_2) = \infty.$$

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